Navier-Stokes equation in Elmer

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Heat transfer modeling in Elmer

When solving for the fluid flow for velocity $\vec{v}$ and pressure p Elmer can account for a large number of different phenomena

- Steady-state flow problems (assuming that steady-state solution exists)
- Transient flow problems
- Incompressible and compresible fluid flow
- Non-newtonian viscosity models
- Solution in deforming domain (ALE formulation)
- Different boundary conditions: given velocity (Dirichlet), traction (Neumann), slip coefficient (Robin),...

For large Reynolds number turbulence models are often a necessity but here they are omitted in this presentation. Tectonic flows often have small Reynolds numbers.

Navier-Stokes equation in Elmer

The incompressible and Newtoanin fluid the Navier-Stokes equation yields,

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right)-\nabla\cdot(2\mu\bar{\bar{\varepsilon}})+\nabla p = \rho\vec{g}, \qquad (1)$$

$$\nabla\cdot\vec{u} = 0. \qquad (2)$$

where ρ is density, μ is the viscosity, $\vec{u}$ is the velocity, p is the pressure and $\bar{\bar{\varepsilon}}$ the linearized strain rate tensor, i.e.

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \qquad (3)$$

The source term $\rho\vec{g}$ usually represents a force due to gravity.

Thermal incompressible flow

For thermal incompressible fluid flows we use the Boussinesq approximation. This means that the temperature only causes an addiotional boyancy force depending on the temperature difference

$$\rho = \rho_0(1 - \beta(T - T_0)), \tag{4}$$

where β is the volume expansion coefficient and the subscript 0 refers to a reference state. Assuming that the gravitational acceleration $\vec{g}$ is the only external force, then the force $\rho_0\vec{g}(1 - \beta(T - T_0))$ is caused in the fluid by temperature variations. This phenomenon is called Grashof convection or natural convection.

Non-newtonian viscosity models

There are several non-newtonian material models. All are functions of the strainrate $\dot{\gamma}$.

Power law

$$\eta = \begin{cases} \eta_0 \dot{\gamma}^{n-1} & \text{if } \dot{\gamma} > \dot{\gamma}_0, \\ \eta_0 \dot{\gamma}_0^{n-1} & \text{if } \dot{\gamma} \leq \dot{\gamma}_0. \end{cases} \tag{5}$$

Carreau-Yasuda

$$\eta = \eta_\infty + \Delta\eta \left(1 + (c\dot{\gamma})^y\right)^{\frac{n-1}{y}}, \tag{6}$$

Cross

$$\eta = \eta_\infty + \frac{\Delta\eta}{1 + c\dot{\gamma}^n}, \tag{7}$$

Powell-Eyring

$$\eta = \eta_\infty + \Delta\eta \frac{\operatorname{asinh}(c\dot{\gamma})}{c\dot{\gamma}}. \tag{8}$$

Temperature development viscosity models

All the viscosity models in Elmer can be made temperature dependent. The current choice is a temperature-dependent viscosity of the form is to multiply the suggested viscosity

$$\eta = \eta_0 \exp(d(1/(T_o + T) - 1/T_r)) \tag{9}$$

where d is the exponential factor, T_o is temperature offset (to allow using of Celcius), and T_r the reference temperature for which the factor becomes one.
Also other types of temperature dependent viscosity models are of course possible using UDFs and MATC expressions.

Navier-Stokes equation in moving mesh

For problems involving deformations the transient Navier-Stokes equation must be solved using Arbitrary Lagrangian-Eulerian (ALE) frame of reference.
Assume that the mesh velocity during the nonlinear iteration is $\vec{c}$. Then the convective term yields

$$((\vec{u} - \vec{c}) \cdot \nabla)\, \vec{u} \approx ((\vec{\mathcal{U}} - \vec{c}) \cdot \nabla)\vec{u}. \tag{10}$$

The additional term including the mesh velocity is the same for both Picard iteration and Newton type of linearization schemes.

Additional body forces

There are many additional body forces that can be accounted for

- Viscous drag that could model flow through porous media
- Additional body forces resulting to moving frame of reference
- Coupling with electrical fields assuming that the fluid is electrically charged
- Coupling with magnetic fields assumimg that the fluid is electrically charged

Boundary conditions

- Dirichlet boundary condition for velocity component u_i is simply

$$u_i = u_i^b. \tag{11}$$

  where value u_i^b can be constant or a function of time, position etc.

- Normal stress may be written in the form

$$\sigma_n = \frac{\gamma}{R} - p_a \tag{12}$$

  where γ is the surface tension coefficient, R the mean curvature and p_a the external pressure.

- One may also give the force vector on a boundary directly as in

$$\overline{\overline{\sigma}} \cdot \vec{n} = \vec{g}. \tag{13}$$

- Tangential stress has the form

$$\vec{\sigma}_\tau = \nabla_s \gamma, \tag{14}$$

where ∇_s is the surface gradient operator. The coefficient γ may be approximated from

$$\gamma = \gamma_0(1 - \vartheta(T - T_0)), \tag{15}$$

where ϑ is the temperature coefficient of the surface tension and the subscript 0 refers to a reference state. Now boundary condition for tangential stress becomes

$$\vec{\sigma}_\tau = -\vartheta\gamma_0 \nabla_s T. \tag{16}$$